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STABILIZATION OF DYNAMICAL SYSTEMS WITH GEOMETRICALLY CONSTRAINED CONTROL*

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A control system, subject to a positional control obtained by "cutoff" of the output delivered by a linear regulator based on the magnitude of geometrical constraints, is investigated for asymptotic stability. The size of the domain of attraction of the equilibrium position of a dynamical system with "cutoff" control is estimated. Necessary and sufficient conditions are derived under which the domain may be made as large as desired by suitable choice of the parameters of a quadratic integral functional.

The design of a time-optimal constrained positional (and even programmed) control which will steer a dynamical system to the origin becomes very difficult in systems with many dimensions /1/. Many authors have therefore proposed non-optimal control laws, which nevertheless enable the problem to be solved within an acceptable time /2-4/. Fairly sophisticated methods are now available for the design of optimal analytical regulators /1, 5-7/; these methods guarantee asymptotic stability of the system in a given position /5-10/. The control - a linear function of the phase coordinates with a constant feedback matrix - is determined by minimizing a certain quadratic integral functional. However, this control may not satisfy additionally specified geometrical constraints in a certain domain of phase space. Necessary and sufficient conditions for a positional control to be optimal in a geometrically constrained problem have been developed /5/, but only for systems with relatively few dimensions can one actually construct the control. A common solution to this problem is to use "cutoff"-type controls; but then the domain of attraction of the equilibrium state has not been determined in the general case. The stability of non-linear control systems with a fixed type of non-linearity has been analysed /8, 9/.

1. Statement of the problem. Consider a completely controllable /1/ dynamical system

$$\dot{x} = Fx + Gu, \quad \operatorname{rank} \| G, FG, F^2G, \ldots, F^{n-1}G \| = n \tag{1.1}$$

Here $x \in R^n$ is the phase vector, F and G are constant matrices, $u \in R^m$ the vector of controls. The initial position x_0 of system (1.1) at time $t = 0$ is given. A positional control under which the trivial solution $x(t) = 0$ of system (1.1) is asymptotically stable in the large /11/ is found by minimizing a certain quadratic functional:

$$J(x_0) = \min_u \int_0^\infty (x^T Ax + u^T Bu)\, dt \tag{1.2}$$

where A and B are constant positive definite symmetric matrices.

It is known /10/ that $J(x_0)$ is a quadratic function of the initial phase state

$$J(x_0) = x_0^T S x_0$$

where S is a symmetric positive definite matrix satisfying the algebraic Riccati equation

$$SRS = SF + F^T S + A, \quad R \equiv GB^{-1}G^T \tag{1.3}$$

It follows from the results in /6, 7/ that in the case of a completely controllable system (1.1) the Riccati equation is always solvable, and there is in fact an algorithm which computes the solution /7/. An optimal control minimizing (1.2) is defined by the following formula /1/:

$$u(x) = -B^{-1}G^T Sx \tag{1.4}$$

Let us assume that the control in system (1.1) is subjected to an additional restriction

$$|u(t)| \leqslant 1 \tag{1.5}$$

Consider the control

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$$u^*(x) = -B^{-1}G^TSx \quad \text{if} \quad |B^{-1}G^TSx| \leqslant 1$$
$$u^*(x) = -B^{-1}G^TSx\,|B^{-1}G^TSx|^{-1} \quad \text{if} \quad |B^{-1}G^TSx| > 1 \tag{1.6}$$

The reader may convince himself that controls of this type need not be optimal for problems involving a functional (1.2) and a geometrical constraint (1.5). Hence such controls do not even guarantee convergence of the integral (1.2) or, consequently, asymptotic stability in the large of the control system (1.1), (1.6). Our aim in this paper is to work out an estimate of the domain of attraction /11/ of the trivial solution $x(t) = 0$ of system (1.1) with cutoff control (1.6) and to determine the conditions under which the size of this domain may be increased without limit by suitable choice of the parameters in the functional (1.2).

2. *Main results.* Suppose we are given two quadratic forms, defined by the matrices L_1 and L_2. We shall say that $L_2 < L_1$, if $x^TL_2x < x^TL_1x$ for all x.

Lemma 1. Consider the control system (1.1) with functional (1.2). Let $S_{1(2)}$ be the solution of the Riccati Eq.(1.3) corresponding to the matrix $A_{1(2)}$. If $A_1 > A_2$, then $S_1 > S_2$.

Proof. Let $u_{1(2)}(t)$ be a programmed control of system (1.1) minimizing the functional (1.2) for the quadratic form with matrix $A_{1(2)}$, and a $x_{1(2)}(t)$ the corresponding trajectory of the system, $x_{1(2)}(0) = x_0$. The assertion of the lemma follows from the following chain of relations:

$$x_0^TS_1x_0 = \int_0^\infty (x_1^TA_1x_1 + u_1^TBu_1)\,dt > \int_0^\infty (x_1^TA_2x_1 + u_1^TBu_1)\,dt \geqslant$$
$$\int_0^\infty (x_2^TA_2x_2 + u_2^TBu_2)\,dt = x_0^TS_2x_0$$

For another proof of this lemma see /12/.

Lemma 2. Let S be the solution of the Riccati Eq.(1.3) corresponding to system (1.1). Then

$$\lim S = 0 \quad \text{as} \quad A \to 0 \tag{2.1}$$

if and only if, for all eigenvalues λ_i of the operator F in (1.1), it is true that $\operatorname{Re}\lambda_i \leqslant 0$.

Proof. Sufficiency. Throughout the proof, $\lim S$ will be considered only as $A \to 0$. In our notation for the limit or the expression $S \to 0$ we shall therefore always omit the condition $A \to 0$. Let θ be a non-singular complex matrix. We shall use the following notation:

$$x_N = \theta^{-1}x, \quad F_N = \theta^{-1}F\theta, \quad S_N = \theta^*S\theta,$$
$$A_N = \theta^*A\theta, \quad G_N = \theta^{-1}G, \quad R_N = \theta^{-1}GB^{-1}G^T\theta^{-1*}$$

and choose θ so that the matrix F_N is in Jordan normal form (the asterisk denotes transposition plus complex conjugation). Eqs.(1.1) remain formally the same written in terms of the variables x_N, F_N, G_N. Let us assume that the matrix F_N has r Jordan blocks along the principal diagonal, the l-th block $(l = 1, \ldots, r)$ being an $n_l \times n_l$ matrix of the form

$$F_l = \begin{Vmatrix} \lambda_l & 1 & 0 & \ldots & 0 \\ 0 & \lambda_l & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_l \end{Vmatrix} = \lambda_l E_{n_l} + I^1_{n_l} \tag{2.2}$$

where E_{n_l} is the $n_l \times n_l$ identity matrix and $I^1_{n_l}$ the $n_l \times n_l$ matrix with ones above the principal diagonal and zeros everywhere else. Let L_{ij} denote the ij element of a matrix L. Thus

$$(I^1_{n_l})_{ij} = 1 \quad \text{if} \quad i = j - 1, \; (I^1_{n_l})_{ij} = 0 \quad \text{if} \quad i \neq j - 1$$
$$(I^{1*}_{n_l})_{ij} = 1 \quad \text{if} \quad j = i - 1, \quad (I^{1*}_{n_l})_{ij} = 0 \quad \text{if} \quad j \neq i - 1 \tag{2.3}$$

We first make the following observation. Let $i_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha_{k(\alpha)}}$ be the indices of the rows in F_N corresponding to the last rows of the Jordan blocks (2.2) with eigenvalues λ_α and $k(\alpha)$ the number of these blocks. Let G_N^α be the submatrix of G_N formed by the rows with these same indices $i_{\alpha_1}, i_{\alpha_2}, \ldots, i_{\alpha_{k(\alpha)}}$. It follows from the results of /13/ that the system is completely controllable if and only if

$$\operatorname{rank} G_N^{\alpha} = k(\alpha), \quad \alpha = 1, 2, \ldots \tag{2.4}$$

In terms of the variables S_N, A_N, R_N, the Riccati Eq.(1.3) takes the following form (from now on, throughout this proof, we omit the index "N"):

$$SRS = F^*S + SF + A, \quad R \equiv GB^{-1}G^* \tag{2.5}$$

Here S, A, R are selfadjoint matrices; S and A are positive definite and R is positive semidefinite.

We begin with the simplest case of F, proving (2.1) on the assumption that $\operatorname{Re}\lambda_l \leqslant 0$ for any l. Let $r=2$ and $n_1 = n_2, l = 1, 2, 2n_1 = n$. Thus, F has two complex-conjugate Jordan blocks along its principal diagonal, belonging to eigenvalues λ_1 and $\lambda_2 = \bar{\lambda}_1$. Let S_i be the i-th column of S. Then, using (2.3), we can write out the matrix Eq.(2.5) for the ij-th element of the matrices on its left and right:

$$S_i^* RS_j = S_{ij}(\lambda_p + \bar{\lambda}_q) + S_{ij-1} + S_{i-1j} + A_{ij} \tag{2.6}$$

Note that when $1 \leqslant i, j \leqslant n_1$ we must take $p = q = 1$ and omit any terms S_{i0} and S_{0j} that may appear on the right. When $n_1 + 1 \leqslant i, j \leqslant n$ we must take $p = q = 2$ and omit any terms $S_{n_1 j}, S_{in_1}$ on the right. When $1 \leqslant i \leqslant n_1, n_1 + 1 \leqslant j \leqslant n$ we take $p = 2, q = 1$ and omit terms S_{in_1}, S_{0j}, if they appear second or third on the right of (2.6).

By assumption, $\operatorname{Re}\lambda_1 \leqslant 0$. There are two possibilities: $\operatorname{Re}\lambda_1 < 0$ and $\operatorname{Re}\lambda_1 = 0$. Let $\operatorname{Re}\lambda_1 < 0$. Then it follows from Eq.(2.6) with $i = j = 1$ that

$$S_1^* RS_1 = 2S_{11} \operatorname{Re}\lambda_1 + A_{11}$$

where R is a positive semidefinite matrix and $\operatorname{Re}\lambda_1 < 0$. Hence, as $A \to 0$ we have $S_{11} \to 0$, and by Sylvester's criterion for S to be positive definite and the fact that this matrix is bounded (see Lemma 1), we have $S_{1j}, S_{j1} \to 0$. When $i, j = 2$ Eq.(2.6) gives

$$S_2^* RS_2 + 2S_{22}(-\operatorname{Re}\lambda_1) = S_{21} + S_{12} + A_{22}$$

As the right-hand side of this equality tends to zero and each term here is non-negative, each term on the left tends to zero, i.e., $S_{22} \to 0$ and $S_{2j}, S_{j2} \to 0$, and so on for all indices $i, j = 3; i, j = 4; \ldots; i, j = n$. Note that we have proved the equality $S_{ii} \to 0$ for any index i such that the corresponding eigenvalue has a negative real part.

Let $\operatorname{Re}\lambda_1 = 0$ and $\operatorname{Im}\lambda_1 \neq 0$. Eq.(2.6), written out for $i = 1$, has the form $(S_{10} = 0)$

$$S_1^* RS_j = S_{1j-1} + A_{1j}, \quad 1 \leqslant j \leqslant n_1 \tag{2.7}$$

This equation with $j = 1$ gives $S_1^* RS_1 \to 0$. Hence, since SRS is positive semidefinite and S is bounded, we obtain $S_1^* RS_j \to 0, j = 1, 2, \ldots, n$. Eq.(2.7) with $j = 2$ gives $S_{11} \to 0$ and

$$S_{1j}, S_{j1} \to 0, \quad j = 1, 2, \ldots, n \tag{2.8}$$

since S is positive definite and bounded, Eq.(2.6) with $i = 2$ gives the following equation (S_{20} must be omitted - see the explanations to formula (2.6)):

$$S_2^* RS_j = S_{2j-1} + S_{1j} + A_{2j}, \quad j = 1, 2, \ldots, n_1$$

Hence, putting $j = 2$ and using (2.8), we obtain $S_2^* RS_2 \to 0$, and so

$$S_2^* RS_j \to 0, \quad j = 1, 2, \ldots, n \tag{2.9}$$

Eq.(2.6) with $i = 2, j = 3$ gives

$$S_2^* RS_3 = S_{22} + S_{13} + A_{23}$$

Hence, by (2.8), (2.9), it follows that $S_{22} \to 0$ and

$$S_{2j} \to 0, \quad j = 1, 2, \ldots, n$$

Reasoning in this way for $i, j = 2, 3, 4, \ldots, n_1$, we conclude that for $i \leqslant n_1 - 1 \cup j \leqslant n_1 - 1$

$$S_{ij} \to 0 \tag{2.10}$$

For $i, j = n_1$, we deduce from Eq.(2.6) that

$$S_{n_1}^* RS_{n_1} = S_{n_1 n_1 - 1} + S_{n_1 - 1 n_1} + A_{n_1 n_1}$$

By (2.10), this equation implies $S_{n_1}^* RS_{n_1} \to 0$, and so

$$S_i^* RS_j \to 0, \quad i \leqslant n_1 \cup j \leqslant n_1 \tag{2.11}$$

Our proof of (2.10) and (2.11) has used only the following properties of the left-hand side of system (2.6): the fact that SRS is bounded and positive semidefinite, and the fact that S is bounded and positive definite. These properties follow from the definition of S (1.2) and Lemma 1. The group of Eqs.(2.6), considered for $1 \leqslant i, j \leqslant n_1$ "links up" with Eqs.(2.6) for $n_1 + 1 \leqslant i, j \leqslant n$ only through the left-hand side; but the above properties of the left-hand side are independent of the number of Jordan blocks. Similar arguments will therefore prove that if $n_1 + 1 \leqslant i, j \leqslant n$, then $S_i^* RS_j \to 0$. Together with (2.11), that gives $SRS \to 0$. Recalling that B is positive definite and using the definition of R (2.5), we finally obtain

$$SG \to 0 \tag{2.12}$$

Reasoning as in the proof of (2.10), we see that for $i \neq n_1, n \cap j \neq n_1, n$

$$S_{ij} \to 0 \tag{2.13}$$

Putting $i = n_1$ and $j = n$, we deduce from (2.6) that

$$S_{n_1}{}^* R S_n = S_{n_1 n}(\lambda_1 + \bar{\lambda}_2) + S_{n_1 n-1} + S_{n_1-1 n} + A_{n_1 n} \tag{2.14}$$

The left-hand side of this equality goes to zero because of (2.12); $S_{n_1 n-1}, S_{n_1-1n} \to 0$ by formula (2.13). By assumption, $\lambda_1 + \bar{\lambda}_2 \neq 0$. Therefore,

$$S_{nn_1}, S_{n_1 n} \to 0 \tag{2.15}$$

Using formulae (2.13) and (2.15), we can now write out (2.12) for the n_1-th row, and the result is

$$S_{n_1 n_1} \| G_{n_1 1}, G_{n_1 2}, \ldots, G_{n_1 m} \| \to 0$$

and similarly for the n-th row. By condition (2.4), there is at least one non-zero component in the n_1-th row of the control matrix G. Therefore $S_{n_1 n_1}$ and, similarly, $S_{nn} \to 0$. We have thus proved (2.1) for $\operatorname{Re} \lambda_1 = 0$, $\operatorname{Im} \lambda_1 \neq 0$.

Let $\operatorname{Im} \lambda_1 = \operatorname{Re} \lambda_1 = 0$, i.e., we have to Jordan blocks with zero eigenvalues. In that case (2.12), (2.13) remain valid. It follows from (2.12), in view of (2.13), that

$$\left\| \begin{matrix} S_{n_1 n_1} & S_{n_1 n} \\ S_{nn_1} & S_{nn} \end{matrix} \right\| \left\| \begin{matrix} G_{n_1 1} & \ldots & G_{n_1 m} \\ G_{n1} & \ldots & G_{nm} \end{matrix} \right\| \to 0 \tag{2.16}$$

By the complete controllability condition (2.4), the second matrix in this product is of rank two, hence the first matrix must also tend to zero. Together with condition (2.13), this proves formula (2.1) for a matrix F with two Jordan blocks.

The idea of the proof for an arbitrary matrix F is exactly the same. No further arguments are necessary to establish formulae like (2.12) and (2.13). A formula of type (2.15) is clearly valid for any indices such that the sum of eigenvalues in parentheses in (2.14) does not vanish. Suppose, then, that the sum in question does vanish. Then, since $\operatorname{Re} \lambda_l \leqslant 0$, $l = 1, 2, \ldots, r$, this means that the relevant indices i, j correspond to the last rows of Jordan blocks (2.2) with identical pure imaginary eigenvalues. We now use the complete controllability condition (2.4) and (2.12). These conditions imply an expression of type (2.16), with the indices n_1, n in the first matrix replaced by the indices of the last rows of the Jordan blocks belonging to the eigenvalues in question. The number of columns in this matrix equals the number of appropriate Jordan blocks and is equal to the rank of the second matrix in (2.16), by the complete controllability condition (2.4). Therefore all the elements of the first matrix tend to zero. Together with the formula of type (2.13) for the other indices, this completes the "sufficiency" part of the proof of Lemma 2.

Necessity. Suppose that for some i we have $\operatorname{Re} \lambda_i > 0$. We assert that then S cannot be made as small as desired by choosing A (1.2). Indeed, otherwise, (2.1) would again by true. The stabilizing control is related to S by formula (1.4). This means that for any fixed initial position x^0 the optimal control $u(t)$ (in the sense of the criterion (1.2)) may be made as small as desired uniformly with respect to all $t \in [0, \infty)$. Let the matrix F in (1.1) be brought to Jordan normal form by a non-singular transformation and suppose that the equation of motion for some k, corresponding to the last row of a Jordan block (2.2) with $\operatorname{Re} \lambda > 0$, is

$$x_k^{\cdot} = \lambda x_k + (Gu)_k, \quad \operatorname{Re} \lambda > 0 \tag{2.17}$$

Let us assume, for simplicity, that u is a scalar and G a vector. Then, by the complete controllability condition (2.4) for system (1.1), we have $G_k \neq 0$. Using Cauchy's formula to solve Eq.(2.17) (omitting the index k from now on), we have

$$x(t) = x^\circ \exp(\lambda t) + \int_0^t \exp(\lambda \tau) G u(\tau)\, d\tau$$

Separating the real and imaginary parts, we obtain

$$x_1(t) = \exp(\lambda_1 t)(x_1^\circ \cos \lambda_2 t - x_2^\circ \sin \lambda_2 t) + \int_0^t \exp(\lambda_1 \tau)(g_1 \cos \lambda_2 \tau - g_2 \sin \lambda_2 \tau)\, u(\tau)\, d\tau$$

$$\lambda = \lambda_1 + i\lambda_2, \quad G = g_1 + ig_2, \quad x = x_1 + ix_2$$

It follows from this equality that if

$$\lambda_1 [x_1^{\circ 2} + x_2^{\circ 2}]^{1/2} > [g_1^2 + g_2^2]^{1/2}$$

then there exists no control $u(\tau)$ which, on the one hand, guarantees that the solution of Eq.(2.17) will tend asymptotically to zero and, on the other, satisfies the constraint $|u(\tau)| \leqslant 1$ for all τ. Thus, if $\operatorname{Re} \lambda_i > 0$ the stabilizing control cannot be made as small as desired, contrary to our assumption that the matrix S can be reduced as much as desired.

Lemma 3. The set of initial values

$$|x_0| \leqslant 2\|B^{-1}G^T\|^{-1}\|S\|^{-1} \tag{2.18}$$

belongs to the domain of attraction of the equilibrium position O of system (1.1) with control (1.6).

Proof. Consider the form $(x^TSx)^{\cdot}$ along trajectories of system (1.1) under the control (1.6). If $|B^{-1}G^TSx| \leqslant 1$, then $(x^TSx)^{\cdot} < 0$ by the definition of S (1.2).

Let $|B^{-1}G^TSx| > 1$. After some simple reduction using Eq.(1.3), we obtain

$$(x^TSx)^{\cdot} = x^T(SGB^{-1}G^TS - A - 2SGB^{-1}G^TS\,|B^{-1}G^TSx|^{-1})\,x$$

It is obvious that a sufficient condition for the quadratic form on the right of this equality to be negative definite on trajectories of system (1.1) with control (1.6) is that

$$2 \geqslant |B^{-1}G^TSx| \tag{2.19}$$

Let x_0 be the position of system (1.1) at time t_0. We have $x^TSx < x_0^TSx_0$ at $t > t_0$ on solutions of system (1.1), because of the inequality $(x^TSx)^{\cdot} < 0$. It can be shown that

$$\max_x |Sx| = (\|S\|\, x_0^TSx_0)^{1/2}, \quad x^TSx \leqslant x_0^TSx_0$$

Thus inequality (2.18) is a sufficient condition for inequality (2.19) to hold.

Note that the estimate $|x_0| \leqslant \|B^{-1}G^T\|^{-1}\|S\|^{-1}$ for the domain of attraction of the equilibrium position O of system (1.1) follows immediately from (1.4) and (1.5).

By Lemmas 1 and 2, the domain (2.18) may be made as large as desired provided that the condition $\operatorname{Re}\lambda_i \leqslant 0$ holds for all eigenvalues of F. Hence we obtain the following theorem.

Let system (1.1) be completely controllable and let $\operatorname{Re}\lambda_i \leqslant 0$ for all its eigenvalues. Then for any bounded set of initial positions x_0 there exists a matrix A of the functional (1.2) guaranteeing that the set will be a subset of the domain of attraction of the equilibrium position O of system (1.1) with control (1.6).

Remark. Let $\operatorname{Re}\lambda_k > 0$ for some k. In the necessity part of the proof of Lemma 2 it was shown that then, for any positional control $u(x)$ satisfying the constraint (1.5), the phase space will contain points that do not belong to the domain of attraction of the equilibrium position O of the system.

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